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# *The Double-six Configuration Connected with the Cubic Surface, and a Related Group of Cremona Transformations.\**

BY EDWARD KASNER.

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The double-six configuration, consisting of two sets of six lines each so related that any line of either set intersects all except a corresponding line of the other set, first presented itself to Schläefli † in the study of the twenty-seven lines upon the cubic surface. In this paper the configuration is investigated in itself, i. e., independently of the cubic surface determined by it, the latter being introduced only incidentally in the final section. The starting point is the theory of five collinear lines. Denoting these by  $L_1, L_2, L_3, L_4, L_5$  and their common tractor by  $M_0$ , then each quadruple as  $L_2, L_3, L_4, L_5$  has (in addition to  $M_0$ ) a proper tractor as  $M_1$ ; thus, five new lines  $M_1, M_2, M_3, M_4, M_5$  are obtained. That these derived lines are themselves collinear, having a common tractor  $L_0$ , was observed incidentally by Schläefli ‡ and verified by Cayley || The proof given in §4 is direct and simple. The twelve lines  $L_i, M_i$ , form a double-six.

The relations between the anharmonic ratios of the thirty points  $P_{ik}$  and

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\* Read in different form before the American Mathematical Society, February 23, 1901.

† “An attempt to determine the twenty-seven lines upon a surface of third order, and to divide such surfaces into species in reference to the reality of the lines upon the surface” (Quarterly Journal of Mathematics, Vol. II, 1858, pp. 55-65, 110-120).  
‡ L. c., p. 117.

|| “On the Double-Sixers of a Cubic Surface” (Collected Papers, Vol. VII, pp. 316-330; Quarterly Journal of Mathematics, Vol. X, 1870, pp. 58-71). Cf. also “On Dr. Wiener’s Model of a Cubic Surface with 27 Real Lines; and on the Construction of a Double-Sixer” (Collected Papers, Vol. VIII, pp. 366-384; Cambridge Philosophical Transactions, Vol. XII, Part I, 1873, pp. 366-383).

thirty planes  $\Pi_{ik}$ ,\* determined by the twelve lines, are discussed in §6; all the ratios are expressible rationally in terms of a fundamental set of four (§7). The Cremona group discussed in §8 arises from the transformations which are induced in the fundamental set by permutations of the lines. In §§9, 10, certain results concerning the double-six and the general cubic surface due to Schur† and Reye‡ are presented from a more simple point of view by employing the relations between the anharmonic ratios.

§1.—*The Coordinate System.* The quintuple§ of collinear lines contains 19 arbitrary constants; but, by properly choosing the system of coordinates, these may be reduced to four projectively essential constants. For this purpose take, as the fundamental tetrahedron of the system, that determined by the points  $P_{25}$ ,  $P_{15}$ ,  $P_{20}$ ,  $P_{10}$ , so that the coordinates of these points are

$$1, 0, 0, 0; 0, 1, 0, 0; 0, 0, 1, 0; 0, 0, 0, 1$$

respectively. The unit point is still at our disposal and may be chosen so as to satisfy any three conditions. Since  $P_{35}$  is collinear with  $P_{15}$  and  $P_{25}$ , its coordinates are of the form  $x_1, x_2, 0, 0$ ; and similarly, those of  $P_{30}$  are of the form  $0, 0, x_3, x_4$ . If, then, we take the coordinates of these points to be 1, 1, 0, 0 and 0, 0, 1, 1 respectively, we, in effect, impose only two conditions upon the unit point. Consider, finally, the plane determined by the line  $L_1$  and by the point of intersection of the line  $L_5$  and the plane  $\Pi_{25}$ ; its coordinates are of the form  $u_1, 0, u_3, 0$ , so that if we take these to be 1, 0, —1, 0 we impose a third condition upon the unit point. The system of coordinates is now completely determined.

§2.—*The Five L Lines.* The four constants which are necessary for the representation of the quintuple may be introduced as follows: The points

\*  $P_{ik}$  is the point of intersection of the lines  $L_i$ ,  $M_k$ ;  $\Pi_{ik}$  is the plane of the same line.

† "Ueber die durch collineare Grundgebilde erzeugten Curven und Flächen" (Mathematische Annalen, Vol. 18, 1881, pp. 1-32).

‡ "Beziehung der allgemeinen Fläche dritter Ordnung zu einer covarianten Fläche dritter Classe" (Mathematische Annalen, Vol. 55, 1901, pp. 257-264).

§ It is assumed throughout the paper that of the five lines constituting the collinear quintuple, no two intersect, and no four have a double tractor, so that the hyperboloid determined by any three does not touch either of the remaining lines.

$P_{40}, P_{50}, P_{45}$  lie on the sides of the fundamental tetrahedron, so that each is representable in terms of a single parameter; the coordinates are of the form  $0, 0, 1, l; 0, 0, 1, m; 1, \lambda, 0, 0$  respectively. The point of intersection of  $L_5$  and  $\Pi_{25}$  lies in the planes  $x_1 - x_3 = 0$  and  $x_4 = 0$ , so that its coordinates are of the form  $1, \mu, 1, 0$ .

The four constants  $l, m, \lambda, \mu$  may be interpreted very simply as anharmonic ratios. The coordinates of the five points of the line  $M_0$  and of the five planes through  $M_0$  are found to be

$$\left. \begin{array}{ll} P_{10}: & 0 \ 0 \ 0 \ 1; \\ P_{20}: & 0 \ 0 \ 1 \ 0; \\ P_{30}: & 0 \ 0 \ 1 \ 1; \\ P_{40}: & 0 \ 0 \ 1 \ l; \\ P_{50}: & 0 \ 0 \ 1 \ m; \end{array} \right\} \quad \left. \begin{array}{ll} \Pi_{10}: & 1 \ 0 \ 0 \ 0; \\ \Pi_{20}: & 0 \ 1 \ 0 \ 0; \\ \Pi_{30}: & -1 \ 1 \ 0 \ 0; \\ \Pi_{40}: & -\lambda \ 1 \ 0 \ 0; \\ \Pi_{50}: & -\mu \ 1 \ 0 \ 0; \end{array} \right\} \quad (1)$$

from which follow the required interpretations:

$$\left. \begin{array}{ll} l = (P_{10}, P_{20}, P_{30}, P_{40}), & \lambda = (\Pi_{10}, \Pi_{20}, \Pi_{30}, \Pi_{40}), \\ m = (P_{10}, P_{20}, P_{30}, P_{50}), & \mu = (\Pi_{10}, \Pi_{20}, \Pi_{30}, \Pi_{50}). \end{array} \right\} \quad (1')$$

The six homogeneous coordinates of each of the lines  $L_1, L_2, L_3, L_4, L_5$  may now be calculated, since upon each we have two known points. Thus the line  $L_4$  passes through the points  $P_{45}$  and  $P_{40}$ ; its coordinates are then the minors of the array

$$\begin{vmatrix} 1 & \lambda & 0 & 0 \\ 0 & 0 & 1 & l \end{vmatrix},$$

or  $p_{12}:p_{13}:p_{14}:p_{34}:p_{42}:p_{23} = 0:1:l:0:-l\lambda:\lambda.$

The table of coordinates is as follows:

$$\left. \begin{array}{ll} L_1: & 0 \ 0 \ 0 \ 0 \ 1 \ 0 \\ L_2: & 0 \ 1 \ 0 \ 0 \ 0 \ 0 \\ L_3: & 0 \ 1 \ 1 \ 0 \ -1 \ 1 \\ L_4: & 0 \ 1 \ l \ 0 \ -l\lambda \ \lambda \\ L_5: & 0 \ 1 \ m \ m \ -m\mu \ \mu \end{array} \right\} \quad (2)$$

The coordinates of the common tractor  $M_0$  are

$$M_0: \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0. \quad (3)$$

§3.—*The Five M Lines.* To exemplify the method, consider the line  $M_4$ , which is the tractor proper to the four lines  $L_1, L_2, L_3, L_5$ . Equating the simultaneous invariants of the intersecting lines to 0, we obtain the conditions:

$$\begin{aligned}(M_4 L_1) &= p_{13} = 0, \\(M_4 L_2) &= p_{42} = 0, \\(M_4 L_3) &= p_{42} + p_{23} - p_{13} + p_{14} = 0, \\(M_4 L_5) &= p_{42} + m p_{23} - m \mu p_{13} + \mu p_{14} = 0,\end{aligned}$$

which, with the quadratic identity

$$(M_4 M_4) = p_{12} p_{34} + p_{13} p_{42} + p_{14} p_{23} = 0,$$

give

$$p_{12} : p_{13} : p_{14} : p_{34} : p_{42} : p_{23} = (m - \mu)^2 : 0 : m(m - \mu) : m^2 : 0 : -m(m - \mu).$$

The coordinates of the remaining lines are found in a similar manner.

$$\left. \begin{array}{llllll} M_1: & A & m(\lambda - l) & \lambda m(\lambda - 1) & \frac{l\lambda m^2(\lambda - 1)(l - 1)}{A} & 0 & -lm(l - 1) \\ M_2: & B & 0 & m(1 - l) & \frac{m^2(\lambda - 1)(l - 1)}{B} & m(l - \lambda) & m(\lambda - 1) \\ M_3: & C & 0 & lm & \frac{l\lambda m^2}{C} & 0 & -\lambda m \\ M_4: & (m - \mu)^2 & 0 & m(m - \mu) & m^2 & 0 & -m(m - \mu) \\ M_5: & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right\} (4)$$

The quantities  $A, B, C$  are functions of  $l, m, \lambda, \mu$ :

$$\left. \begin{aligned} A &= m(1 - l)(\mu - \lambda) - \mu(1 - \lambda)(m - l), \\ B &= (1 - \lambda)(m - l) - (1 - l)(\mu - \lambda), \\ C &= \lambda m - l\mu. \end{aligned} \right\} (5)$$

§4.—*Schlaefli's Theorem.* Five lines in general determine a linear complex. Let the equation of the linear complex to which  $M_1, M_2, M_3, M_4, M_5$  belong be

$$D_{12} p_{12} + D_{13} p_{13} + D_{14} p_{14} + D_{34} p_{34} + D_{42} p_{42} + D_{23} p_{23} = 0,$$

then the coefficients  $D_{ik}$  are proportional to the determinants of the fifth order formed from the matrix whose rows are the coordinates of five lines  $M_1, \dots, M_5$ .

After some reduction, these are found to be

$$\begin{aligned} D_{12} &= 0, & D_{13} &= l\lambda m^2\mu B^2, & D_{14} &= \lambda m^2 AB, \\ D_{34} &= (m - \mu) ABC, & D_{42} &= -mA^2, & D_{23} &= lm\mu AB. \end{aligned}$$

Substituting these values, we find

$$D_{12} D_{34} + D_{13} D_{42} + D_{14} D_{23} = 0,$$

i. e., the invariant of the complex vanishes. From a known theorem, then, the complex is special, and the lines belonging to it are all collinear.

*The lines  $M_1, M_2, M_3, M_4, M_5$  obtained from five collinear lines  $L_1, L_2, L_3, L_4, L_5$  by taking the tractor proper to each set of four of the latter lines are also collinear.*

The coordinates of the common tractor  $L_0$  of the  $M$  lines are

$$L_0: (m - \mu) ABC, \quad -mA^2, \quad lm\mu AB, \quad 0, \quad l\lambda m^2\mu B, \quad \lambda m^2 AB. \quad (6)$$

§5.—*The Double-six Configuration  $L, M, P, \Pi$ .* From the original set of five collinear lines  $L_1, L_2, L_3, L_4, L_5$ , there is thus derived a configuration of twelve lines  $L_0, L_1, L_2, L_3, L_4, L_5, M_0, M_1, M_2, M_3, M_4, M_5$ . Each  $L$  line intersects all the  $M$  lines except the one whose index is the same, and similarly, each  $M$  line intersects five of the  $L$  lines. The various pairs of intersecting lines determine 30 points and 30 planes

$$P_{i\kappa}, \Pi_{i\kappa}, \quad (i, \kappa = 0, 1, 2, 3, 4, 5; i \neq \kappa)$$

where  $P_{i\kappa}$  denotes the point of intersection of  $L_i$  and  $M_\kappa$ , and  $\Pi_{i\kappa}$  denotes the plane of the same two lines.

In the following, indicate by  $\iota_0, \iota_1, \iota_2, \iota_3, \iota_4, \iota_5$  any permutation of the six indices 0, 1, 2, 3, 4, 5. Through each point  $P_{\iota_0 \iota_1}$  there pass 9 planes

$$\Pi_{\iota_0 \iota_1}, \Pi_{\iota_0 \iota_2}, \Pi_{\iota_2 \iota_1}, \Pi_{\iota_0 \iota_3}, \Pi_{\iota_3 \iota_1}, \Pi_{\iota_0 \iota_4}, \Pi_{\iota_4 \iota_1}, \Pi_{\iota_0 \iota_5}, \Pi_{\iota_5 \iota_1};$$

and similarly, in each plane there lie 9 points. On each  $L$  line  $L_{\iota_0}$  there are five points  $P_{\iota_0 \iota_1}, P_{\iota_0 \iota_2}, P_{\iota_0 \iota_3}, P_{\iota_0 \iota_4}, P_{\iota_0 \iota_5}$ , and through it there pass five planes  $\Pi_{\iota_0 \iota_1}, \Pi_{\iota_0 \iota_2}, \Pi_{\iota_0 \iota_3}, \Pi_{\iota_0 \iota_4}, \Pi_{\iota_0 \iota_5}$ ; similarly with respect to the  $M$  lines. The 30 points lie by fives in 6 lines in two distinct ways, and the thirty planes pass by fives through 6 lines in two ways. Any five  $L$  lines, as well as any five  $M$  lines, are

collinear; the common tractor of  $L_{i_0} L_{i_1} L_{i_2} L_{i_3} L_{i_4}$  is  $M_{i_5}$ . In a certain sense, the configuration is closed; no new lines are obtained by taking the tractors of four  $L$  or four  $M$  lines, for the two tractors of  $L_{i_0}, L_{i_1}, L_{i_2}, L_{i_3}$  are  $M_{i_4}$  and  $M_{i_5}$  and the two tractors of  $M_{i_0}, M_{i_1}, M_{i_2}, M_{i_3}$  are  $L_{i_4}$  and  $L_{i_5}$ .

The configuration was derived from  $L_1, L_2, L_3, L_4, L_5$ , but it is just as well determined by any five  $L$  lines or any five  $M$  lines. In particular, it is seen that the relation between the original quintuple  $L_1, \dots, L_5$  and the derived quintuple  $M_1, \dots, M_5$  is a reciprocal one; just as the latter lines are the tractors of the former taken in fours, so the former are the tractors of the latter.

We may state the results obtained as follows:

*In connection with any five collinear lines  $L_1, L_2, L_3, L_4, L_5$ , there exists a covariant line  $L_0$ , uniquely defined by the fact that it is collinear with any four of the five, but not with all five. The relation between the six lines  $L_0, L_1, L_2, L_3, L_4, L_5$  is a symmetrical one, i. e., the covariant line of any five is the sixth. There presents itself, then, a conjugate set of six lines  $M_0, \dots, M_5$ , standing in the same symmetrical relation; these lines are the tractors of  $L$  lines taken by fives. The relation between the conjugate sets is of involutory character.*

§6.—*Relations between the Anharmonic Ratios.* Consider the points in which  $L_1$  is intersected by the lines  $M_0, M_2, M_3, M_4, M_5$ . The coordinates of the point of intersection of any two intersecting lines  $p_{i\kappa}, p'_{i\kappa}$  are proportional to the determinants formed from the array

$$\begin{vmatrix} 0 & p_{34} & p_{42} & p_{23} \\ -p_{42} & -p_{14} & 0 & p_{12} \\ -p'_{42} & -p'_{14} & 0 & p'_{12} \end{vmatrix}.$$

Thus the coordinates of the point  $P_{13}$ , the intersection of the lines  $L_1$  and  $M_3$ , are, by (2) and (3), the determinants formed from the array

$$\begin{vmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & lm & 0 & C \end{vmatrix},$$

which reduce to 0,  $C$ , 0,  $lm$  respectively.

By this method the following table of coordinates is obtained :

$$\begin{array}{lclcl} P_{10}: & 0 & 0 & 0 & 1 \\ P_{12}: & 0 & B & 0 & m(1-l), \\ P_{13}: & 0 & C & 0 & lm, \\ P_{14}: & 0 & m-\mu & 0 & m, \\ P_{15}: & 0 & 1 & 0 & 0. \end{array}$$

The anharmonic ratios of these collinear points are equal to the anharmonic ratios of the corresponding parameters

$$\infty, \quad \frac{m(1-l)}{B}, \quad \frac{lm}{C}, \quad \frac{m}{m-\mu}, \quad 0. \quad (7)$$

Calculating the ratio of the last four points, we find

$$(P_{12}, P_{13}, P_{14}, P_{15}) = \frac{l(1-m)}{m(1-l)}$$

But from (1),

$$(P_{20}, P_{30}, P_{40}, P_{50}) = \frac{l(1-m)}{m(1-l)}.$$

Therefore

$$(P_{12}, P_{13}, P_{14}, P_{15}) = (P_{20}, P_{30}, P_{40}, P_{50}),$$

or employing a more abbreviated notation\* for the anharmonic ratios,

$$L_1(2\ 3\ 4\ 5) = M_0(2\ 3\ 4\ 5).$$

From the symmetry of the configuration, we have then the result :

$$L_{i_1}(i_2\ i_3\ i_4\ i_5) = M_{i_0}(i_2\ i_3\ i_4\ i_5), \quad (8)$$

which may be translated as follows :

*The anharmonic ratio of the points in which any four of five collinear lines cut the common tractor of all five is equal to the anharmonic ratio of the points on the fifth line, which are collinear with three of the four lines.*

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\* The anharmonic ratio of the four points in which the lines  $M_{i_2}$ ,  $M_{i_3}$ ,  $M_{i_4}$ ,  $M_{i_5}$  intersect the line  $L$ , is thus denoted by  $L_{i_1}(i_2\ i_3\ i_4\ i_5)$ ; while that of the four planes determined by the same lines is denoted by  $L_{i_1}(i_2\ i_3\ i_4\ i_5)$ .

Another relation may be obtained by combining the above with the well-known theorem concerning the two tractors of four lines; the four points of either tractor and the four planes of the other tractor have the same anharmonic ratio. In the present case, consider the four lines  $L_{i_2}, L_{i_3}, L_{i_4}, L_{i_5}$ ; their tractors are  $M_{i_0}, M_{i_1}$ , so that

$$M_{i_0}(i_2 i_3 i_4 i_5) = \bar{M}_{i_1}(i_2 i_3 i_4 i_5). \quad (9)$$

From (8) we have then

$$L_{i_1}(i_2 i_3 i_4 i_5) = \bar{M}_{i_1}(i_2 i_3 i_4 i_5), \quad (10)$$

which may be expressed:

*The anharmonic ratio of the four points on one of five collinear lines which are collinear with three of the remaining lines is equal to the anharmonic ratio of the four planes determined by these remaining lines and their common tractor.*

Another relation is obtained by considering the anharmonic ratio of the points  $P_{10}, P_{13}, P_{14}, P_{15}$ , which, by means of (1), is found to be

$$L_1(0\ 3\ 4\ 5) = \frac{l(m-\mu)}{m(l-\lambda)} = \frac{1-\mu/m}{1-\lambda/l}.$$

Introducing the interpretations of  $l, m, \lambda, \mu$  from (1), we have

$$L_1(0\ 3\ 4\ 5) = \frac{1 - \frac{\bar{M}_0(1\ 2\ 3\ 5)}{\bar{M}_0(1\ 2\ 3\ 5)}}{1 - \frac{\bar{M}_0(1\ 2\ 3\ 4)}{\bar{M}_0(1\ 2\ 3\ 4)}},$$

or, by permuting the indices,

$$L_{i_1}(i_2 i_3 i_4 i_5) = \frac{1 - \frac{\bar{M}_{i_2}(i_1 i_0 i_3 i_5)}{\bar{M}_{i_2}(i_1 i_0 i_3 i_5)}}{1 - \frac{\bar{M}_{i_2}(i_1 i_0 i_3 i_4)}{\bar{M}_{i_2}(i_1 i_0 i_3 i_4)}}. \quad (11)$$

*The complete system of relations between the anharmonic ratios of the configuration consists of the fundamental relations (8), (10) and (11), together with the well-known relations\* between the anharmonic ratios of five collinear points or planes.*

In the first place, by means of (8), (10), (11), it is possible to express the ratios of four points or planes of one line in terms of the ratios of the points

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\* See, for example, M. J. M. Hill, "The Anharmonic Ratios of the Roots of a Quintic" (Proceedings of the London Mathematical Society, Vol. XIV, p. 182).

and planes of any other line. Thus:

$$\begin{aligned}
 L_1(2\ 3\ 4\ 5) &= M_0(2\ 3\ 4\ 5) &= \bar{L}_0(2\ 3\ 4\ 5) \\
 &= \bar{M}_1(2\ 3\ 4\ 5) &= L_1(2\ 3\ 4\ 5) \\
 &= \frac{1 - \frac{\bar{M}_2(1\ 0\ 3\ 5)}{\bar{M}_2(1\ 0\ 3\ 5)}}{1 - \frac{\bar{M}_2(1\ 0\ 3\ 4)}{\bar{M}_2(1\ 0\ 3\ 4)}} &= \frac{1 - \frac{L_2(1\ 0\ 3\ 5)}{\bar{L}_2(1\ 0\ 3\ 5)}}{1 - \frac{L_2(1\ 0\ 3\ 5)}{\bar{L}_2(1\ 0\ 3\ 5)}} \\
 &= \frac{1 - \frac{\bar{M}_3(1\ 0\ 2\ 4)}{\bar{M}_3(1\ 0\ 2\ 4)}}{1 - \frac{\bar{M}_3(1\ 0\ 2\ 5)}{\bar{M}_3(1\ 0\ 2\ 5)}} &= \frac{1 - \frac{L_3(1\ 0\ 2\ 4)}{\bar{L}_3(1\ 0\ 2\ 4)}}{1 - \frac{L_3(1\ 0\ 2\ 5)}{\bar{L}_3(1\ 0\ 2\ 5)}} \\
 &= \frac{1 - \frac{\bar{M}_4(1\ 0\ 5\ 3)}{\bar{M}_4(1\ 0\ 5\ 3)}}{1 - \frac{\bar{M}_4(1\ 0\ 5\ 2)}{\bar{M}_4(1\ 0\ 5\ 2)}} &= \frac{1 - \frac{L_4(1\ 0\ 5\ 3)}{\bar{L}_4(1\ 0\ 5\ 3)}}{1 - \frac{L_4(1\ 0\ 5\ 2)}{\bar{L}_4(1\ 0\ 5\ 2)}} \\
 &= \frac{1 - \frac{\bar{M}_5(1\ 0\ 4\ 2)}{\bar{M}_5(1\ 0\ 4\ 2)}}{1 - \frac{\bar{M}_5(1\ 0\ 4\ 3)}{\bar{M}_5(1\ 0\ 4\ 3)}} &= \frac{1 - \frac{L_5(1\ 0\ 4\ 2)}{\bar{L}_5(1\ 0\ 4\ 2)}}{1 - \frac{L_5(1\ 0\ 4\ 3)}{\bar{L}_5(1\ 0\ 4\ 3)}}
 \end{aligned}$$

From this it follows that all the ratios may be expressed in terms of those connected with any one line, say  $M_0$ . But the ratios of the five points on  $M_0$  are rationally expressible in terms of two  $l$  and  $m$ ; and similarly, the ratios of the five planes through  $M_0$  are expressible rationally in terms of two  $\lambda$  and  $\mu$ . The four ratios  $l, m, \lambda, \mu$  are independent, so that there can be no relations in addition to those enumerated in the above theorem.

#### §7.—The Complete System of Anharmonic Ratios.

*Of the anharmonic ratios connected with the double-six, only four are independent. For a fundamental set of independent ratios we may take*

$$\begin{aligned}
 l &= M_0(1\ 2\ 3\ 4), & m &= M_0(1\ 2\ 3\ 5), \\
 \lambda &= \bar{M}_0(1\ 2\ 3\ 4), & \mu &= \bar{M}_0(1\ 2\ 3\ 5).
 \end{aligned} \tag{12}$$

*All the ratios are expressible rationally in terms of these by functions of at most the fourth degree.*

The following table\* gives the anharmonic ratio of any quadruple of points situated on an  $L$  line. The ratios of any quadruple of points on an  $M$  line, or of any quadruple of planes, may also be found in the same table, in virtue of the equalities (8) and (10).

$L_0(1\ 2\ 3\ 4) = \lambda,$	$L_1(0\ 2\ 3\ 4) = \frac{(m-1)C}{(m-l)(m-\mu)},$
$L_0(1\ 2\ 3\ 5) = \mu,$	$L_1(0\ 2\ 3\ 5) = \frac{(1-l)C}{(l-\lambda)(l-m)},$
$L_0(1\ 2\ 4\ 5) = \frac{\mu}{\lambda},$	$L_1(0\ 2\ 4\ 5) = \frac{(l-1)(m-\mu)}{(m-1)(l-\lambda)},$
$L_0(1\ 3\ 4\ 5) = \frac{1-\mu}{1-\lambda},$	$L_1(0\ 3\ 4\ 5) = \frac{l(m-\mu)}{m(l-\lambda)},$
$L_0(2\ 3\ 4\ 5) = \frac{\lambda(1-\mu)}{\mu(1-\lambda)},$	$L_1(2\ 3\ 4\ 5) = \frac{l(m-1)}{m(l-1)},$
$L_2(1\ 0\ 3\ 4) = \frac{\lambda(m-l)(m-\mu)}{(m-l)C},$	$L_3(1\ 2\ 0\ 4) = -\frac{A}{mB},$
$L_2(1\ 0\ 3\ 5) = \frac{\mu(l-\lambda)(m-l)}{(l-1)C},$	$L_3(1\ 2\ 0\ 5) = -\frac{A}{lB},$
$L_2(1\ 0\ 4\ 5) = \frac{\mu(l-\lambda)(m-1)}{\lambda(m-\mu)(l-1)},$	$L_3(1\ 2\ 4\ 5) = \frac{m}{l},$
$L_2(1\ 3\ 4\ 5) = \frac{m-1}{l-1},$	$L_3(1\ 0\ 4\ 5) = \frac{m(l-\lambda)(1-\mu)}{l(m-\mu)(1-\lambda)},$
$L_2(0\ 3\ 4\ 5) = \frac{\lambda(\mu-m)}{\mu(\lambda-l)},$	$L_3(2\ 0\ 4\ 5) = \frac{(l-\lambda)(1-\mu)}{(m-\mu)(1-\lambda)},$
$L_4(1\ 2\ 3\ 0) = -\frac{\lambda m B}{A},$	$L_5(1\ 2\ 3\ 4) = l,$
$L_4(1\ 2\ 3\ 5) = m,$	$L_5(1\ 2\ 3\ 0) = -\frac{l\mu B}{A},$
$L_4(1\ 2\ 0\ 5) = -\frac{A}{\lambda B},$	$L_5(1\ 2\ 4\ 0) = -\frac{\mu B}{A},$
$L_4(1\ 3\ 0\ 5) = \frac{A}{(\lambda-1)C},$	$L_5(1\ 3\ 4\ 0) = \frac{(\mu-1)C}{A},$
$L_4(2\ 3\ 0\ 5) = \frac{\lambda B}{(1-\lambda)C},$	$L_5(2\ 3\ 4\ 0) = \frac{(1-\mu)C}{\mu B}.$

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\*  $A, B, C$  are the functions of  $l, m, \lambda, \mu$  defined in (5).

§8.—*The Cremona Group*  $G_{720}$ . If we denote by  $i_0 i_1 i_2 i_3 i_4 i_5$  any permutation of the indices, the anharmonic ratios which correspond to the fundamental set (12) are

$$\begin{aligned} l_i &= M_{i_0} (i_1 i_2 i_3 i_4), & m_i &= M_{i_0} (i_1 i_2 i_3 i_5), \\ \lambda_i &= \bar{M}_{i_0} (i_1 i_2 i_3 i_4), & \mu_i &= \bar{M}_{i_0} (i_1 i_2 i_3 i_5), \end{aligned}$$

But from §7 these are rational functions of  $l, m, \lambda, \mu$ :

$$\begin{aligned} l_i &= f_i (l, m, \lambda, \mu), & m_i &= \phi_i (l, m, \lambda, \mu), \\ \lambda_i &= \psi_i (l, m, \lambda, \mu), & \mu_i &= \chi_i (l, m, \lambda, \mu), \end{aligned}$$

where  $f_i, \psi_i, \phi_i, \chi_i$  denote rational (in general fractional) functions of the fourth or lower degree. Thus to every literal substitution of the six indices

$$S_i: \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ i_0 & i_1 & i_2 & i_3 & i_4 & i_5 \end{pmatrix} \quad (13)$$

corresponds a rational transformation of the four fundamental invariants

$$T_i: \begin{aligned} l' &= l_i = f_i (l, m, \lambda, \mu), & m' &= m_i = \phi_i (l, m, \lambda, \mu), \\ \lambda' &= \lambda_i = \psi_i (l, m, \lambda, \mu), & \mu' &= \mu_i = \chi_i (l, m, \lambda, \mu). \end{aligned} \quad (14)$$

It is now to be shown that the system of transformations  $T$  so obtained form a group.

Consider any two substitutions  $S_i$  and  $S_j$ :

$$S_j: \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ j_0 & j_1 & j_2 & j_3 & j_4 & j_5 \end{pmatrix} = \begin{pmatrix} i_0 & i_1 & i_2 & i_3 & i_4 & i_5 \\ k_0 & k_1 & k_2 & k_3 & k_4 & k_5 \end{pmatrix},$$

their product  $S_k$ , and the corresponding transformations  $T_i, T_j, T_k$ . Then we have, in the first place,

$$l_k = f_k (l, m, \lambda, \mu), \text{ etc.,}$$

and in the second place, since the permutation  $k_0 \dots k_5$  is obtained from  $i_0 \dots i_5$  by the substitution  $S_j$ ,

$$l^k = f_j (l_i, m_i, \lambda_i, \mu_i) = f_j (f_i, \phi_i, \psi_i, \chi_i), \text{ etc.}$$

Therefore,

$$\begin{aligned} f_j (f_i, \phi_i, \psi_i, \chi_i) &= f_k, & \phi_j (f_i, \phi_i, \psi_i, \chi_i) &= \phi_k, \\ \chi_j (f_i, \phi_i, \psi_i, \chi_i) &= \chi_k, & \chi_j (f_i, \phi_i, \psi_i, \chi_i) &= \chi_k, \end{aligned}$$

which expresses the essential group property of the system of transformations  $T$ ,

$$T_i T_j = T_k.$$

In particular, it follows that the inverse of  $T_i$  is also a transformation of the set, and therefore the transformations are birational or Cremona transformations.

*The transformations  $T$ , induced by the permutations of the indices, constitute a group  $G_{720}$  of Cremona transformations which is holoedrically isomorphic with the symmetric substitution group on six letters.*

A set of generators may be obtained by writing down the transformations which correspond to generators of the symmetric group  $S(0\ 1\ 2\ 3\ 4\ 5)$ . Thus the following five form a convenient set of generators:

$$\begin{array}{llll} (34) \sim T_1: & \frac{1}{l} & \frac{m}{l} & \frac{1}{\lambda} \quad \frac{\mu}{\lambda} \\ (23)(45) \sim T_2: & 1-m & 1-l & 1-\mu \quad 1-\lambda \\ (45) \sim T_3: & m & l & \mu \quad \lambda \\ (12) \sim T_4: & \frac{1}{l} & \frac{1}{m} & \frac{1}{\lambda} \quad \frac{1}{\mu} \\ (01) \sim T_5: & \frac{(\mu-1)(\lambda m-l\mu)}{(\mu-\lambda)(m-\mu)} & \frac{(\lambda-1)(\lambda m-l\mu)}{(\mu-\lambda)(l-\lambda)} & \frac{(m-1)(\lambda m-l\mu)}{(m-l)(m-\mu)} \quad \frac{(l-1)(\lambda m-l\mu)}{(m-l)(l-\lambda)} \end{array}$$

The subgroup  $G_{120}$ , consisting of the transformations corresponding to those permutations which leave the index 0 unaltered, is essentially a group in two variables, since the generators  $T_1, T_2, T_3, T_4$  transform  $l, m$  and  $\lambda, \mu$  cogrediently. The group in two variables so obtained is identical with the cross-ratio group discussed by E. H. Moore\* and H. E. Slaught.† The total group  $G_{720}$ , however, cannot be expressed in two variables. Expressed in homogeneous form, five variables are necessary,

$$t_1 : t_2 : t_3 : t_4 : t_5 = \lambda : \mu : l : m : 1.$$

\* "The Cross-Ratio Group of  $n!$  Cremona Transformations of Order  $n-3$  in Flat Space of  $n-3$  Dimensions" (American Journal of Mathematics, Vol. XXII, 1900, p. 279).

† "The Cross-Ratio Group of 120 Quadratic Cremona Transformations of the Plane" (American Journal of Mathematics, Vol. XXII, 1900, pp. 343-380). The generators  $K, L, M, T'$ , given on p. 344, correspond to the generators  $T_1, T_2, T_3, T_4$  employed above.

The generators then become

$$\begin{array}{lllll}
 T_1: & t_3 t_5 & t_2 t_3 & t_1 t_5 & t_1 t_4 & t_1 t_3 \\
 T_2: & t_5 - t_2 & t_5 - t_1 & t_5 - t_4 & t_5 - t_2 & t_5 \\
 T_3: & t_2 & t_1 & t_2 & t_3 & t_5 \\
 T_4: & t_2 t_3 t_4 t_5 & t_1 t_3 t_4 t_5 & t_1 t_2 t_4 t_5 & t_1 t_2 t_3 t_5 & t_1 t_2 t_3 t_4 \\
 T_5: & \left\{ \begin{array}{ll} (t_2 - t_1)(t_3 - t_1)(t_4 - t_5)(t_1 t_4 - t_2 t_3) & (t_2 - t_1)(t_4 - t_2)(t_3 - t_5)(t_1 t_4 - t_2 t_3) \\ (t_4 - t_3)(t_3 - t_1)(t_2 - t_5)(t_1 t_4 - t_2 t_3) & (t_4 - t_3)(t_4 - t_3)(t_1 - t_5)(t_1 t_4 - t_2 t_3) \\ t_5(t_2 - t_1)(t_4 - t_3)(t_3 - t_1)(t_4 - t_2). \end{array} \right. & & & & 
 \end{array}$$

§9.—*The Automorphic Correlation  $\Omega$ .*\* Consider two quintuples of collinear lines  $L_1, L_2, L_3, L_4, L_5$  and  $L'_1, L'_2, L'_3, L'_4, L'_5$ . Denote the common tractor of the first set by  $M_0$ , the point of intersection of  $M_0$  and  $L_i$  by  $P_i$ , the plane of the same pair of lines by  $\Pi_i$ , with a similar notation for the second set. We now prove the following

LEMMA. *The two quintuples are homographic when*

$$\begin{aligned}
 (P_1 P_2 P_3 P_4 P_5) \oslash (P'_1 P'_2 P'_3 P'_4 P'_5), \\
 (\Pi_1 \Pi_2 \Pi_3 \Pi_4 \Pi_5) \oslash (\Pi'_1 \Pi'_2 \Pi'_3 \Pi'_4 \Pi'_5);
 \end{aligned}$$

*they are correlative when*

$$\begin{aligned}
 (P_1 P_2 P_3 P_4 P_5) \oslash (\Pi'_1 \Pi'_2 \Pi'_3 \Pi'_4 \Pi'_5), \\
 (\Pi_1 \Pi_2 \Pi_3 \Pi_4 \Pi_5) \oslash (P'_1 P'_2 P'_3 P'_4 P'_5).
 \end{aligned}$$

It will be sufficient to give the proof of the second part, since that of the first follows the same scheme. Assuming the second set of relations, it is to be shown that there is a correlation transforming the first quintuple into the second. The general space correlation, or reciprocity, contains 15 parameters. The number of correlations transforming  $M_0$  into  $M'_0$  is therefore  $\infty^{11}$ . If we impose the

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\* Schur, l. c., p. 12.

conditions that  $\Pi'_1, \Pi'_2, \Pi'_3$  shall correspond to  $P_1, P_2, P_3$ , and that  $\Pi'_1, \Pi'_2, \Pi'_3$  shall correspond to  $P_1, P_2, P_3$ , the number of correlations is reduced to  $\infty^5$ . Each of those correlations, in virtue of the assumed relations, will then transform  $P_4, P_5, \Pi_4, \Pi_5$  into  $\Pi'_4, \Pi'_5, P'_4, P'_5$ , and, therefore, any line of the first quintuple  $L_i$  is transformed into a line belonging to the pencil determined by  $P'_4$  and  $\Pi'_5$ . If, then, we require the correlation to transform  $L_1, \dots, L_5$  into  $L'_1, \dots, L'_5$ , we impose just five additional (linear) conditions and the correlation is completely determined. Similarly, it may be proved that when the first set of relations holds, there is a definite homography transforming the first quintuple into the second.

We apply the lemma to the quintuples  $M_1, \dots, M_5$  and  $L_1, \dots, L_5$  with the common tractors  $L_0$  and  $M_0$  respectively. From the relations (10) it follows that the conditions of the second part of the lemma are fulfilled, so that there exists a correlation  $\Omega$  transforming  $M_1 \dots M_5$  into  $L_1 \dots L_5$ .

*The quintuple of lines  $M_1, \dots, M_5$ , obtained from a quintuple of collinear lines  $L_1, \dots, L_5$  by taking the tractor proper to each set of four of the latter lines, is correlative to the original quintuple.*

This is an extension of Schlaefli's theorem, the latter merely states that  $M_1, \dots, M_5$  are collinear, while the above includes in addition the relations (10).

Consider now the effect of the correlation  $\Omega$  upon the double-six. The common tractor  $L_0$  of the one quintuple becomes the common tractor  $M_0$  of the other. Taking the tractor proper to each set of four, it follows that  $L_1, \dots, L_5$  transform into  $M_1, \dots, M_5$  and, therefore,  $M_0$  into  $L_0$ . Hence  $P_{ik}$  becomes  $\Pi_{ki}$  and  $\Pi_{ik}$  becomes  $P_{ki}$ .

*There exists for the double-six configuration an automorphic correlation  $\Omega$  which interchanges the lines  $L_i, M_i$  and the points and planes  $P_{ik}, \Pi_{ki}$ .*

The character of  $\Omega$  is determined as follows: Since  $\Omega$  transforms  $P_{ik}$  into  $\Pi_{ki}$  and  $\Pi_{ki}$  into  $P_{ik}$ , the collineation  $\Omega^2$  leaves the points  $P_{ik}$  invariant; in particular, the five points  $P_{10}, P_{20}, P_{13}, P_{23}, P_{45}$ , no four of which are coplanar, are invariant, and, therefore,  $\Omega^2$  is the identical transformation, i. e.,  $\Omega$  is involutorial. There are, however, two species of involutorial transformations in space of three dimensions, namely: polarities, in which corresponding points and planes are conjugate with respect to a proper quadric surface; and null-systems, in

which all corresponding points and planes are incident. The latter species is at once excluded in the present case by the fact that the plane  $\Pi_{ki}$  corresponding to the point  $P_{ik}$  does not pass through it. Therefore,

*The automorphic correlation  $\Omega$  of the double-six is a polarity, i e., there exists a proper quadric surface  $Q$  with respect to which the points  $P_{ik}$  and the plane  $\Pi_{ki}$  are conjugate.*

§10 — *The Cubic Surfaces  $F$  and  $\Phi$ .*\* The number of arbitrary constants involved in a set of five collinear lines is 19, which is equal to the number of constants in a quaternary cubic form. In fact, it is easy to show that through five collinear lines  $L_1 \dots L_5$  there pass a single surface  $F$  of the order, and also a single surface  $\Phi$  of the third class. Thus a surface of the third order is determined by the 19 points,  $P_{10}, P_{12}, P_{13}, P_{14}, P_{20}, P_{21}, P_{23}, P_{24}, P_{30}, P_{31}, P_{32}, P_{34}, P_{40}, P_{41}, P_{42}, P_{43}, P_{51}, P_{52}, P_{53}$ ; but the first four points are on  $M_0$ , therefore, the entire line  $M_0$  lies in the surface, and in particular the point  $P_{50}$ ; then each of the lines  $L_1 \dots L_5$  has four points in the surface, and so lies in it. The surfaces  $F$  and  $\Phi$  pass through the 12 lines of the double-six, as may be proved by showing that each line has four points in common with  $F$  and four planes in common with  $\Phi$ .

It is well known† that the 27 lines on  $F$  consist of the 12 lines  $L_i, M_i$  and the 15 lines  $c_{ik} = c_{ki}$  obtained as the intersections of the pairs of planes  $\Pi_{ik}, \Pi_{ki}$ . From the principle of duality, it follows that the 27 lines on  $\Phi$  consist of the 12 lines  $L_i, M_i$  and the 15 lines  $d_{ik} = d_{ki}$  obtained by joining the pairs of points  $P_{ik}, P_{ki}$ .

The relation between  $F$  and  $\Phi$  is obtained by considering the polarity  $\Omega$  defined in the previous section:  $\Omega$  transforms  $F$  into a surface of the third class passing through the double-six; but there is only one such surface, namely,  $\Phi$ ; therefore,  $\Omega$  transforms  $F$  into  $\Phi$ , and similarly,  $\Phi$  is transformed into  $F$ .

*Through five collinear lines there pass a single surface  $F$  of the third order, and a single surface  $\Phi$  of the third class. These surfaces intersect in the double-six determined by the five lines, and are reciprocally related by the polarity  $\Omega$ .*‡

\* Reye, l. c.

† Schläefli, l. c.

‡ Compare the corresponding theorem with regard to three arbitrary lines: Through three lines there pass a single surface of the second order and a single surface of the second class; these two surfaces coincide.

The number of double-six's which can be formed from the 27 lines on a cubic surface is 36.\* The preceding theorem thus determines, for the general cubic surface  $F$  of third order, a set of 36 irrational quadric covariants  $[Q]$ , and a corresponding set of 36 cubic contravariants  $[\Phi]$ .

COLUMBIA UNIVERSITY, NEW YORK, *November*, 1901.

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\* Schläefli, l. c., p. 115.